Problems for the third week

1. In each of the following problems (a) through (d) find the solution of the given initial value problem and compute $\lim_{t\to\infty} y(t)$.

   (a) $y'' + 5y' + 6y = 0, \qquad y(0) = 2, y'(0) = 3$
   (b) $y'' + y' - 2y = 0, \qquad y(0) = 1, y'(0) = 1,$
   (c) $y'' + 4y' + 3y = 0, \qquad y(0) = 2, y'(0) = -1,$
   (d) $y'' + 8y' - 9y = 0, \qquad y(1) = 1, y'(1) = 0.$

2. Consider the initial value problem

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, y'(0) = \beta,$$

   where $\beta > 0$.

   (a) Solve the initial value problem.
   (b) Determine the coordinates (t_m, y_m) of the maximum point of the solution as functions of β
   (c) Determine the smallest value of β for which $y_m \geq 4$.
   (d) Determine the behavior of t_m and y_m as $\beta \to \infty$.

3. In each problems (a) through (d) use Euler's formula to write the given expression in the form $a + ib$.

   (a) e^{-3+6i}
   (b) e^{1+2i}
   (c) $e^{i\pi}$
   (d) 2^{1-i}

4. In each of the following problems (a) through (d) find the solution of the given initial value problem

   (a) $16y'' - 8y' + 145y = 0, \qquad y(0) = -2, y'(0) = 1$
   (b) $y'' + 4y = 0, \qquad y(0) = 1, y'(0) = 1,$
   (c) $y'' - 2y' + 5y = 0, \qquad y(\pi/2) = 0, y'(\pi/2) = 2,$

(d) $y'' + 2y' + 2y = 0, \qquad y(\pi/4) = 2, y'(\pi/4) = -2.$

5. In each of the following problems (a) through (d) find the solution of the given initial value problem

   (a) $y'' - y' + 0.25y = 0, \qquad y(0) = 2, y'(0) = \frac{1}{3}$
   (b) $9y'' - 12y' + 4y = 0, \qquad y(0) = 2, y'(0) = -1$
   (c) $9y'' + 6y' + 82y = 0, \qquad y(0) = -1, y'(0) = 2$
   (d) $y'' + 4y' + 4y = 0, \qquad y(-1) = 2, y'(-1) = 1$

6. Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \qquad y(0) = 1, y'(0) = -4.$$

   (a) Solve the initial value problem and plot its solution for $0 \leq t \leq 5$.
   (b) Determine where the solution has the value zero.
   (c) Determine the coordinates (t_0, y_0) of the minimum points.
   (d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b. Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

Results for all exercises and fully worked out solutions (for 1. (a), 3.(a), 4.(a), 5.(a))

1. (a) The characteristic equation is:

$$r^2 + 5r + 6 = 0.$$

The roots are: $r_1 = -2, r_2 = -3$. Thus the general solution is:

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}. \tag{1}$$

The derivative of the general solution is:

$$y' = -2c_1 \mathrm{e}^{-2t} - 3c_2 \mathrm{e}^{-3t}. \tag{2}$$

Substitute the initial conditions into equations (1) and (2) to get

$$
\begin{aligned}
2 = y(0) &= c_1 + c_2 \\
3 = y'(0) &= -2c_1 - 3c_2
\end{aligned}
$$

Solving this we obtain:

$$c_1 = 9, \quad c_2 = -7$$

We substitute these into (1) and get the solution of the initial value problem:

$$y = 9\mathrm{e}^{-2t} - 7\mathrm{e}^{-3t}.$$

1. (b) $y = \mathrm{e}^t$, $\lim\limits_{t\to\infty} y(t) = \infty$,

1. (c) $y = \frac{5}{2}\mathrm{e}^{-t} - \frac{1}{2}\mathrm{e}^{-3t}$, $\lim\limits_{t\to\infty} y(t) = 0$.

1. (d) $y = \frac{1}{10}\mathrm{e}^{-9(t-1)} + \frac{9}{10}\mathrm{e}^{t-1}$, $\lim\limits_{t\to\infty} y(t) = \infty$.

2. (a) $y = (6+\beta)\mathrm{e}^{-2t} - (4+\beta)\mathrm{e}^{-3t}$,

2. (b) $t_m = \ln\left[(12+3\beta)\,/\,(12+2\beta)\right]$, $y_m \frac{4}{27}(6+\beta)^3/(4+\beta)^2$,

2. (c) $\beta = 6(1+\sqrt{3})$,

2. (d) $t_m \to \ln(3/2)$.

3. (a) $\mathrm{e}^{-3+6i} = \mathrm{e}^{-3}\mathrm{e}^{-6i} = \mathrm{e}^{-3}\left(\cos 6 + i\sin 6\right) \cong 0.0478 - 0.0139\cdot i$.

3. (b) $\mathrm{e}\cos 2 + i\mathrm{e}\sin 2$,

3. (c) -1,

3. (d) $2\cos(\ln 2) - 2i\sin(\ln 2)$.

4. (a) The characteristic equation is:

$$16r^2 - 8r + 145 = 0$$

and its roots are: $r_1 = 1/4 + 3i, r_2 = 1/4 - 3i$. The general solution is

$$y = c_1\mathrm{e}^{t/4}\cos(3t) + c_2\mathrm{e}^{t/4}\sin(3t). \tag{3}$$

The derivative of the general solution is:

$$y' = 1/4\, c_1\, e^{1/4\,t}\cos(3\,t) - 3\, c_1\, e^{1/4\,t}\sin(3\,t) + 1/4\, c_2\, e^{1/4\,t}\sin(3\,t) + 3\, c_2\, e^{1/4\,t}\cos(3\,t) \tag{4}$$

Now we substitute the initial values into the equations (3) and (4) and get

$$
\begin{aligned}
-2 = y(0) &= c_1 \\
1 = y'(0) &= \frac{1}{4}c_1 + 3c_2.
\end{aligned}
$$

Thus we obtain $c_1 = -2$ and $c_2 = 1/2$. We substitute these into (3) to get the solution of the initial value problem:

$$y = -2\mathrm{e}^{t/4}\cos(3t) + \frac{1}{2}\mathrm{e}^{t/4}\sin(3t).$$

4. (b) $y = \frac{1}{2}\sin(2t)$
4. (c) $y = -\mathrm{e}^{t-\pi/2}\sin(2t)$
4. (d) $y = \sqrt{2}\mathrm{e}^{-(t-\pi/4)}\cos t + \sqrt{2}\mathrm{e}^{-(t-\pi/4)}\sin t$

5. (a) The characteristic equation is:

$$r^2 - r + 0.25 = 0$$

and the roots are $r := r_1 = r_2 = 1/2$. The general solution is:

$$y = c_1\mathrm{e}^{t/2} + c_2 \cdot t \cdot \mathrm{e}^{t/2}. \tag{5}$$

The derivative of the general solution is:

$$y' = 1/2 \cdot\ c_1\, e^{1/2\,t} + c_2\, e^{1/2\,t} + 1/2 \cdot\ c_2\, te^{1/2\,t}. \tag{6}$$

Substitute the initial values into equations (5) and (6). We get

$$\begin{aligned} 2 = y(0) &= c_1 \\ \frac{1}{3} = y'(0) &= \frac{1}{2}c_1 + c_2 \end{aligned}$$

Thus $c_1 = 2$ and $c_2 = -\frac{2}{3}$. We substitute these into (5) to get the solution of the initial value problem:

$$y = 2\mathrm{e}^{t/2} - \frac{2}{3} \cdot t \cdot \mathrm{e}^{t/2}.$$

5. (b) $y = 2\mathrm{e}^{2t/3} - \frac{7}{3}t\mathrm{e}^{2t/3}$,
5. (c) $y = -\mathrm{e}^{-t/3}\cos(3t) + \frac{5}{9}\mathrm{e}^{-t/3}\sin(3t)$,
5. (d) $y = 7\mathrm{e}^{-2(t+1)} + 5t\mathrm{e}^{-2(t+1)}$.

6. (a) $\mathrm{e}^{-3t/2} - \frac{5}{2}t\mathrm{e}^{-3t/2}$,
6. (b) $t = 2/5$,
6. (c) $t_0 = 16/15$, $y_0 = -\frac{5}{3}\mathrm{e}^{-8/5}$
6. (d) $y = \mathrm{e}^{-3t/2} + \left(b + \frac{3}{2}\right)t\mathrm{e}^{-3t/2}$, $b = -\frac{3}{2}$.

Problems for the fourth week with results and fully worked out solutions to Problems 1 (a)–(e),4,9

1. Use the method of undetermined coefficients to find the general solution of the following five differential equations:
   (a) $y'' - 3y' - 4y = 3e^{2t}$
   (b) $y'' - 3y' - 4y = 2\sin t$
   (c) $y'' - 3y' - 4y = -8t\cos(2t)$
   (d) $y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8t\cos(2t)$
   (e) $y'' - 3y' - 4y = 2e^{-t}$

2. In each of the following four problems find the general solution of the differential equation
   (a) $y'' - 2y' - 3y = 3e^{2t}$
   (b) $y'' + 2y' + 5y = 3\sin(2t)$
   (c) $y'' - 2y' - 3y = -3te^{-t}$
   (d) $y'' + 2y' + y = 2e^{-t}$

3. Find the solution of the initial value problem:
$$y'' + 4y = t^2 + 3t, \qquad y(0) = 0, y'(0) = 2.$$

4. Find the general solution of
$$y'' + 4y = 3\csc t, \qquad (\csc t = 1/\sin t)$$

5. First use the method of variation of parameters to find the general solution of the following two differential equations. Then use the method of undetermined coefficients to check your answers.
   (a) $y'' - 5y' + 6y = 2e^{t}$.
   (b) $4y'' - 4y' + y = 16e^{t/2}$.

6. Find the general solution of the differential equation
$$y'' + y = \tan t, \qquad 0 < t < \frac{\pi}{2}.$$

7. Find the general solution of the differential equation

$$4y'' + y = 2\sec(t/2), \qquad -\pi < t < \pi \ (\sec t = 1/\cos t).$$

8. Consider the

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \quad t > 0.$$

First verify that the functions

$$Y_1 = t, \qquad Y_2 = te^t$$

form a fundamental solution of the corresponding homogenous equation $t^2y'' - t(t+2)y' + (t+2)y = 0$. Then find the general solution of the inhomogeneous equation.

9. A mass 4 lb stretches a spring 2 in. Supposed that the mass is displaced an additional 6 in (1/2 ft) in the positive direction and then released. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 lb/sec. Formulate the initial value problem that governs the $y(t)$ motion of the mass.

Results

1.Since

$$y_{i,gen} = Y_{h,gen} + y_{i,p} \tag{1}$$

first we solve the homogenous part

$$Y'' - 3Y' - 4Y = 0.$$

of the equation. Since the roots of the characteristic equation $r^2 - 3r - 4$ are

$$r_1 = -1, \qquad r_2 = 4 \tag{2}$$

we obtain that the general solution of the homogenous part of the equation is:

$$Y_{h,alt} = c_1 e^{-t} + c_2 e^{4t}. \tag{3}$$

Observe that this is the same in all the five part of this problem. So, we only need to find a particular solution in each of the next five problems.

1a. Using that 2 is not a root of the characteristic equation we should try to find $y = y_{i,p}$ in the form

$$y = c \cdot \mathrm{e}^{2t}.$$

To find the constant we need to substitute back to the differential equation $y'' - 3y' - 4y = 3\mathrm{e}^{2t}$. For that we need

$$y' = 2c\mathrm{e}^{2t}, \qquad y'' = 4c\mathrm{e}^{2t}.$$

Thus

$$(4c - 3 \cdot 2c - 4 \cdot c) \cdot \mathrm{e}^{2t} = 3\mathrm{e}^{2t}.$$

This yields that $c = -1/2$. That is

$$y = y_{i,p} = -\frac{1}{2}\mathrm{e}^{2t}.$$

This and (3) together implies that

$$y_{i,alt} = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{4t} - \frac{1}{2}\mathrm{e}^{2t}$$

1b. We only need to find $y = y_{i,p}$. We try to find it in the form:

$$y = A\cos t + B\sin t.$$

That is we need to compute A and B such that $y = A\cos t + B\sin t$ is a solution of

$$y'' - 3y' - 4y = 2\sin t. \tag{4}$$

Thus we need

$$y' = -A\sin t + B\cos t, \qquad y'' = -A\cos t - B\sin t.$$

After substituting this to (4) we obtain

$$(-5A - 3B)\cos t + (3A - 5B)\sin t = 2\sin t.$$

The coefficients of $\cos t$ and the coefficients of $\sin t$ must be the same:

$$\begin{aligned} -5A - 3B &= 0 \\ 3A - 5B &= 2 \end{aligned}$$

So, $A = 3/17$ and $B = -5/17$ therefore

$$y_{i,p} = \frac{3}{17}\cos t - \frac{5}{17}\sin t.$$

Using this, (1) and (3) we obtain

$$y_{i,alt} = \underbrace{c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t}}_{Y_{h,alt}} + \underbrace{\frac{3}{17}\cos t - \frac{5}{17}\sin t}_{y_{i,p}}.$$

1c. We need to find A, B such that the function $y_{i,p} = A\mathrm{e}^t\cos(2t) + B\mathrm{e}^t\sin(2t)$ is a solution of $y'' - 3y' - 4y = -8t\cos(2t)$. After differentiating twice:

$$y' = (A + 2B)\,\mathrm{e}^t\cos(2t) + (-2A + B)\,\mathrm{e}^t\sin(2t),$$
$$y'' = (-3A + 4B)\,\mathrm{e}^t\cos(2t) + (-4A - 3B)\,\mathrm{e}^t\sin(2t).$$

Substituting into $y'' - 3y' - 4y = -8t\cos(2t)$ yields

$$\begin{aligned} 10A + 2B &= 8 \\ 2A - 10B &= 0. \end{aligned}$$

The solution is $A = 10/13$ and $B = 2/13$. Thus

$$y_{i,p} = \frac{10}{13}\mathrm{e}^t\cos(2t) + \frac{2}{13}\mathrm{e}^t\sin(2t).$$

Using this, (1) and (3) we obtain

$$y_{i,alt} = \underbrace{c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{4t}}_{Y_{h,alt}} + \underbrace{\frac{10}{13}\mathrm{e}^t\cos(2t) + \frac{2}{13}\mathrm{e}^t\sin(2t)}_{y_{i,p}}.$$

1d. Observe that the right hand side of the equation is just the sum of the right hand sides of the previous three equations and the left hand side is the same. Using that the equations are linear it follows that the we get a particular solution as the sum of the particular solutions in the previous three problems. That is

$$y_{i,p} = -\frac{1}{2}\mathrm{e}^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}\mathrm{e}^t\cos(2t) + \frac{2}{13}\mathrm{e}^t\sin(2t)$$

Thus the general solution is:

$$y_{i,alt} = \underbrace{c_1 e^{-t} + c_2 e^{4t}}_{Y_{h,alt}} + \underbrace{-\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t\cos(2t) + \frac{2}{13}e^t\sin(2t)}_{y_{i,p}}.$$

1e. Using (2) we see that -1 is a simple root of the characteristic polynomial. Thus we try to find particular solution in the form

$$y = (At + B)e^{-t}.$$

To do so, we need to find the constants A and B for which $y = (At+B)e^{-t}$ is a solution of $y'' - 3y' - 4y = 2e^{-t}$. First we compute

$$y' = (A - B)e^{-t} - Ate^{-t}, \qquad y'' = (-2\,A + B)e^{-t} + Ate^{-t}.$$

After substitution we obtain a solution $A = -2/3$, $B = 0$. Thus a particular solution of $y'' - 3y' - 4y = 2e^{-t}$ is

$$y_{i,p} = -\frac{2}{3}te^{-t}.$$

Using this, (1) and (3) we obtain

$$y_{i,gen} = \underbrace{c_1 e^{-t} + c_2 e^{4t}}_{Y_{h,gen}} + \underbrace{-\frac{2}{3}te^{-t}}_{y_{i,p}}.$$

2a. $y = c_1 e^{3t} + c_2 e^{-t} - e^{2t}$
2b. $y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t)$
2c.$y = c_1 e^{3t} + c_2 e^{-t}\frac{3}{16}te^{-t} + \frac{3}{8}t^2 e^{-t}$
2d.$y = c_1 e^{-t} + c_2 te^{-t} + t^2 e^{-t}$.
3. $y = \frac{7}{10}\sin(2t) - \frac{19}{40}\cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$
4. The general solution is given by

$$y_{i,gen} = Y_{h,gen} + y_{i,p} \tag{5}$$

The homogenous part is

$$Y'' + 4Y = 0.$$

The characteristic polynomial of this equation is $r^2 + 4r = 0$. Therefore the roots of the characteristic polynomial are

$$r_1 = 2i, \qquad r_2 = -2i.$$

the general solution of the homogenous part is

$$Y_{h,alt} = c_1 \cos(2t) + c_2 \sin(2t). \tag{6}$$

To find $y = y_{i,p}$ we need to use the Variation of parameters method: That is we need to determine the functions $c_1(t), c_2(t)$ such that

$$y = c_1(t) \cos(2t) + c_2(t) \sin(2t) \tag{7}$$

is a solution of $y'' + 4y = 3 \csc t$. For that we need to solve the system of equations

$$\begin{array}{rcl} c_1'(t) \cos(2t) + c_2'(t) \sin(2t) & = & 0 \\ -2c_1'(t) \sin(2t) + 2c_2'(t) \cos(2t) & = & 3 \csc t. \end{array}$$

From the first equation we get

$$c_2'(t) = -c_1'(t) \frac{\cos(2t)}{\sin(2t)}.$$

Then substituting this into the second equation yields:

$$c_1'(t) = -\frac{3 \csc t \sin(2t)}{2} = -3 \cos t.$$

Substituting this back to the one but last equation we get

$$c_2'(t) = \frac{3}{2} \csc t - 3 \sin t.$$

Now we integrate (without the additive constants since we need only one particular solution) and we get

$$c_1(t) = -3 \sin(t), \;\; c_2(t) = \frac{3}{2} \ln|\csc t - \cot t| + 3 \cos t.$$

Thus

$$y = y_{i,p} = (-3 \sin(t)) \cdot \cos(2t) + (\frac{3}{2} \ln|\csc t - \cot t| + 3 \cos t) \cdot \sin(2t).$$

Using (5) and (6) the general solution is

$$\begin{array}{rcl} y_{i,gen} & = & c_1 \cos(2t) + c_2 \sin(2t) \\ & + & (-3 \sin(t)) \cdot \cos(2t) + (\frac{3}{2} \ln|\csc t - \cot t| + 3 \cos t) \cdot \sin(2t). \end{array}$$

5a. $y = c_1 e^{2t} + c_2 e^{3t} + e^t$.
5b. $y = c_1 e^{t/2} + c_2 t e^{t/2} + 2t^2 e^{t/2}$.
6. $y = c_1 \cos t + c_2 \sin t - (\cos t) \ln(\tan t + \sec t)$.
7. $y = c_1 \cos(t/2) + c_2 \sin(t/2) + t \sin(t/2) + 2 [\ln \cos(t/2)] \cos(t/2)$.
8. $y = c_1 t + c_2 t e^t - 2t^2$.
9. First we want to write down the equation:

$$my''(t) + \gamma y'(t) + ky(t) = F(t),$$

We measure the displacement in ft the mass in lb and the time in sec. We recall that the acceleration due to the gravity is $g = 32 \frac{ft}{sec^2}$. First we observe that there is no external force. So $F(t) = 0$. We determine the mass m from

$$mg = 4\ lb.$$

That is

$$m = \frac{1}{8} \frac{lb \cdot sec^2}{ft}.$$

We determine the damping coefficient γ from the assumption that the damping force $\gamma y' = 6lb$ when $y' = 3ft/sec$. Thus

$$\gamma = \frac{6\ lb}{3\ ft/sec} = 2 \frac{lb \cdot sec}{ft}.$$

The spring constant k is determined from the assumption that the mass stretches the spring by $L = 2in = 1/6ft$. Thus $4lb = m \cdot g = k \cdot L$ yields

$$k = \frac{4\ lb}{1/6ft} = 24 \frac{lb}{ft}.$$

Thus the equation which governs the motion of the motion is:

$$y'' + 16y' + 192y = 0, \qquad y(0) = \frac{1}{2}, y'(0) = 0.$$

Problems and results for the fifth week with fully worked out solutions for the problems: 1,2,3.(a),(b),(c), 6.(a)

1. Solve the following initial value problem:

$$y'' = \frac{1}{\sqrt{1-x^2}}, \qquad y(0) = 3,\ y'(0) = 1$$

2. A rod is loaded by a bending moment that is proportinal to the value $f(x)$ at each coordinate x. It is known that the shape of this rod's median can be computed by solving the following differential equation:

$$\frac{y''}{(1+(y')^2)^{3/2}} = f(x).$$

Determine the shape of the rod if the bending moment follows

$$f(x) = 1 - x$$

and the initial conditions are given by

$$y(0) = y'(0) = 0.$$

3. (a) Consider the differential equation of free mechanical vibration without damping

$$my'' + ky = 0. \tag{1}$$

Solve it as an incomplete second order differential equation.

(b) Solve differential equation (1) as a second order linear equation.

(c) Prove that the solutions you obtained in 3a and 3b are the same.

4. Find the general solution of the following differential equations.

(a)

$$(y')^2 + 2yy'' = 0,$$

(b)

$$y'' = \frac{1}{4\sqrt{y}},$$

(c)
$$yy'' + (y')^2 = 1$$

5. Solve the following second order differential equation:
$$xy'' - y' = x^3.$$

6. Solve the following differential equations:
   (a) $2x\cos y + [2y\cos y - (x^2+y^2)\sin y]\, y' = 0,$
   (b) $xdy + ydx = 0,$
   (c) $\frac{x}{x^2+y^2}y' = \frac{y}{x^2+y^2},$
   (d) $2x(\sin y + 1) + x^2\cos y \cdot y' = 0.$

Results

1. Integrate both sides twice against x:
$$y' = \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C_1,$$
$$y = \int (\arcsin x + C_1)dx = \arcsin x + \sqrt{1-x^2} + C_1 x + C_2.$$

Substitute into the initial conditions
$$\begin{aligned} 3 = y(0) &= 1 + C_2 \\ 1 = y'(0) &= \arcsin 0 + 1. \end{aligned}$$

This yields
$$C_1 = 1 \text{ and } C_2 = 2.$$

The solution of the initial value problem is:
$$y = \arcsin x + \sqrt{1-x^2} + x + 2.$$

2. This equation does not contain y. So, we introduce a new variable $p = y'$. Then
$$p(x) = y'(x) \text{ and } p'(x) = y''(x).$$

We substitute this into our equation and get

$$\int \frac{dp}{(1+p^2)^{3/2}} = \int f(x)dx.$$

Let us write

$$\int f(x)dx = F(x).$$

After integration

$$\frac{p}{\sqrt{1+p^2}} = F(x) + c_1.$$

Then it follows that

$$p(x) = \frac{F(x)+c_1}{\sqrt{1-(F(x)+C-1)^2}}.$$

Using that $y' = p$ yields

$$y = \int \frac{F(x)+c_1}{\sqrt{1-(F(x)+c_1)^2}}dx. \tag{2}$$

In the special case when $f(x) = 1-x$ we get by integration that

$$F(x) + c_1 = x(1-\frac{x}{2}) + c_1.$$

That is

$$y' = p = \frac{x(1-\frac{x}{2})+c_1}{\sqrt{1-\left(x(1-\frac{x}{2})+c_1\right)^2}}.$$

Using the initial value $y'(0) = 0$ we get that

$$c_1 = 0.$$

Now we substitute this into (**??**). Using the other initial value $y(0) = 0$ follows that

$$y(x) = \int_{t=0}^{x} \frac{t(1-\frac{t}{2})}{\sqrt{1-t^2\left(1-\frac{t}{2}\right)^2}}dt.$$

This is an *elliptic integral* so we cannot express as a formula which contains elementary functions. However, we can draw its graph using computer (see Figure **??**).
3a. 1 Using the notation

$$\omega_0 := \sqrt{\frac{k}{m}}$$

our differential equation is

$$y'' = -\omega_0^2 y. \tag{3}$$

Since it does not contain the independent variable x explicitly we can introduce the new variable p as

$$y' = p = p(y), \qquad y'' = \frac{dp}{dy}p,$$

Using this substitution equation (**??**) is:

$$p \cdot p' = -\omega_0^2 y \tag{4}$$

Here we mean $p' = \frac{dp}{dy}$. Then equation (**??**) is a separable differential equation. We multiply by dy which follows

$$p \cdot dp = -\omega_0^2 y \cdot dy.$$

Interating both sides yields

$$\frac{1}{2}p^2 = -\frac{\omega_0^2}{2}y^2 + C_1.$$

That is

$$\frac{dy}{dt} = y' = p = \pm\sqrt{2C_1 - \omega_0^2 y^2}.$$

In this way we obtained the separable differential equation

$$\frac{dy}{dt} = \pm\sqrt{2C_1 - \omega_0^2 y^2}, \tag{5}$$

which is equivalent to

$$\frac{dy}{\sqrt{2C_1 - \omega_0^2 y^2}} = \pm dt.$$

Using that
$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$
we get
$$y = \frac{\sqrt{2C_1}}{\omega_0}\sin(\omega_0(t+C_2)). \tag{6}$$

In the one but last equation both the $+$ and $-$ signs results the same solution. To see this note that if the constants are C_2 and C_2' appropriately then $\omega_0 C_2' = \omega_0 C_2 + \pi$. But this is not important.

3b. Consider equation (1) as a linear equation. Its characteristics polynomial is
$$mr^2 - k = 0.$$
Using the notation $\omega_0 := k/m$ we obtain that the roots are:
$$r_1 = i\cdot\omega_0, \qquad r_2 = i\cdot\omega_0.$$
The general solution is:
$$y = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t). \tag{7}$$

3c. In equation (**??**) on the right hand side we pull out $\sqrt{c_1^2+c_2^2}$-et. Then we get
$$y = \sqrt{c_1^2+c_2^2}\cdot\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos(\omega_0 t) + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin(\omega_0 t)\right). \tag{8}$$
Note that the vector
$$\mathbf{v} = \left(\frac{c_2}{\sqrt{c_1^2+c_2^2}}, \frac{c_1}{\sqrt{c_1^2+c_2^2}}\right)$$
is a unit vector. Thus there is an angle θ such that $\mathbf{v} = (\cos\theta, \sin\theta)$. For this angle θ we have
$$\cos\theta = \frac{c_2}{\sqrt{c_1^2+c_2^2}} \text{ and } \sin\theta = \frac{c_1}{\sqrt{c_1^2+c_2^2}}.$$
Substituting this into equation (**??**) yields:
$$y = \sqrt{c_1^2+c_2^2}\cdot\underbrace{(\sin\theta\cdot\cos(\omega_0 t) + \cos\theta\cdot\sin(\omega_0 t))}_{\sin(\theta+\omega_0 t)}.$$

Using the notation $d_1 := \sqrt{c_1^2 + c_2^2}$ and $d_2 := \theta/\omega_0$ we get

$$y = d_1 \sin(\omega_0(t + d_2)). \tag{9}$$

This gives the same as (**??**) with $d_2 = c_2$ and $d_1 = \frac{\sqrt{2C_1}}{\omega_0}$.

4a. $y = C_1(x + C_2)^{2/3}$,

4b. $3x = 4(\sqrt{y} - 2C - 1)\sqrt{C_1 + \sqrt{y}} + C_2$,

4c. $(x + C_2)^2 - y^2 = C_1$.

5. $y = \frac{x^4}{8} + \frac{C_1 x^2}{2} + C - 2$.

6a. We write $y' = \frac{dy}{dx}$ and multiply both sides with dx.

$$\underbrace{2x \cos y}_{M(x,y)}\, dx + \underbrace{\left[2y \cos y - (x^2 + y^2) \sin y\right]}_{N(x,y)}\, dy = 0$$

Using that

$$\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x} = -2x \sin y$$

we see that the differential equation is exact. That is there exist functions $F : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathbf{grad}(F) = (M, N).$$

That is

$$\frac{\partial F}{\partial x} = M, \qquad \frac{\partial F}{\partial y} = N.$$

Using that $\frac{\partial F}{\partial x} = M$ we can write

$$F(x, y) = \int M(x, y) dx + h(y) = x^2 \cos y + h(y). \tag{10}$$

We obtain the function $h(y)$ from $\frac{\partial F}{\partial y} = N$. Namely,

$$\frac{\partial F}{\partial y} = -x^2 \sin y + h'(y) = \underbrace{2y \cos y - (x^2 + y^2) \sin y}_{N}.$$

This yields

$$h'(y) = 2y \cos y - y^2 \sin y.$$

After integration

$$h(y) = y^2 \cos y + C_1.$$

We substitute this back into (**??**). In this way we get that

$$F(x,y) = (x^2 + y^2)\cos y + C_1.$$

So, the general solution of the differential equation is

$$(x^2 + y^2)\cos y = Const.$$

6b. $xy = Const$

6c. $\arctan \frac{y}{x} = Const.$

6d. $x^2 \sin y + \sin y = Const.$